Analyzing the Optimality Conditions for Nonlinear Semi-Infinite Multi-objective (Pareto) Optimization Problems Involving Generalized Invex Functions

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Abstract

This research investigated the invexity and generalized invexity to analyze the Optimality Conditions for the new classes of nonlinear optimization problems of the so-called nonlinear semi-infinite programming (NSIP). In this study, Fritz-John (FJ) and Karush-Kuhn-Tucker (KKT) type optimality conditions are also established under this invexity. Finally we recall the definition of a class of generalized invex functions and then we apply them to nonlinear semi-infinite multi-objective programming (NSIMP).

Key Words: Nonlinear semi-infinite multi-objective programming, Invexity and generalized invexity, Invex functions, Fritz-John and Karush-Kuhn-Tucker type, Optimality conditions.

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1 Introduction

This paper concerns the study of new classes of nonlinear optimization problems of the so called Nonlinear Semi-infinite Programming (NSIP). This types of problems are generally defined on either infinite dimensional spaces of decision variables or contains infinitely many of equality and inequality constraints with arbitrary (may not be compact) index sets (but not both). So the Semi-infinite programming (SIP) allows us for either infinitely many constraints or infinitely many variables but not both. An Infinite programming (IP) problem can be reduced to semi-infinite programs in case of finite dimensional spaces of decision variables.

In optimization theory, a Semi-infinite programming problem is also an optimization problem which occurs in many branches of science, engineering, and economics as well as in other areas. Due to the growing number of theoretical and practical applications in which constrains are depend on time or space and thus can be formulated as Semi-infinite programming. SIP has recently become one of the most substantial research areas in applied mathematics and operation research [1]. Although the origins of SIP optimization are related to Chebyshev approximation, to the classical work of Haar on linear semi-infinite system [2], and to the Fritz John optimality conditions [3], the term SIP was coined in 1962 by Charnes, Cooper and Kortanek in some papers devoted to LSIP [4,5,6]. But in the mean time G.B.Dantzig (1963) solved the standard convex problems with SIP. Gustafson and Kortanek proposed, during the early 1970s, the first numerical methods for SIP models arising in applications [7]. A large class of engineering design problems such as electronic circuit design, seismic-resistant structure design, and single-input single-output control system design etc can be formulated as nonlinear semi-infinite program (see Polak, 1981). The theory of SIP was developed in the last two decades due to its important applications.

A common approach to SIP which yields the global convergence is the use of SQP techniques in conjunction with an exact penalty function [8,9,10,11]. The methods given in papers [9,10,11] use the implicit function theorem on each semi-infinite constraint to test convergence. The main purpose of this article is to highlight the nonlinear semi-infinite problem and their characteristics on the

connection with optimality or KKT conditions. It is interesting that the semi-infinite programming is an extension of finite nonlinear programming. Many solution methods in finite optimization are based on the KKT system. The invexity and generalized invexity plays a central role in optimization theory and mathematical economics.

2 Theoretical Backgrounds

In this section we consider the following so-called Nonlinear Semi-Infinite Programming problem (NSIP):

(P) Minimize $f(x)$. (2.1)

Subject to, $g(x,u) \le 0$, $(\forall x \in X, u \in U)$.

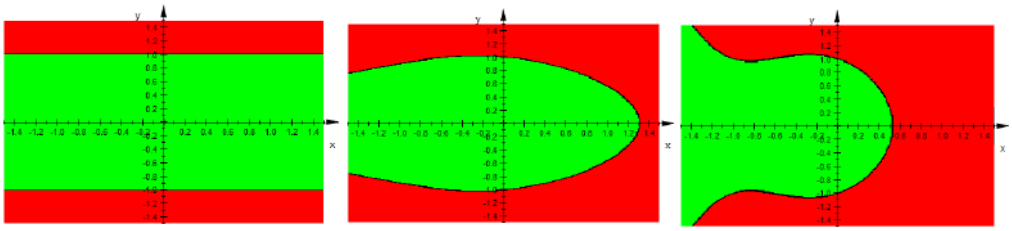
Where, $x \in X \subset R^n$ is a nonempty set and $U \subset R^m$ is an infinite parameter countable set and does not depend on x. Let f can be taken as $f = (f_1, f_2, \ldots . f_p) : R^n \to R^p$ and $g : R^n \times U \to R^p$. This two functions are assumed to be real valued nonlinear (may not both) functions and at least twice continuously differentiable with respect to x and u respectively. We can also define a NSIMP as multi-objective optimization problem (MOP) for given p objective functions which have to be minimized. In that case, the set of optimal compromises with respect to the objective functions is called the Pareto set. Any point $x \in R^n$ in parameter space will be called a Pareto point if there is no other point which is at least as good as x through all the objectives and strictly better in at least one objective functions.

Example 1: Consider the problem like (P), and the following problem satisfies all the conditions of (P): Minimize $z = f(x) = \frac{1}{3}x_1^2 + x_2^2 + \frac{1}{2}x_1$

Subject to, $u^2 x_1 + x_2^2 - (1 - x_1^2 u^2)^2 \ge 0$, $u \in [0,1]$.

This problem holds for all condition of the type of problem (P). In this problem there are finite number of variables and infinite number of constraints

depending on $u \in [0,1]$. With $[x_1^0, x_2^0] = [1,2]$ and optimal solution is $[x_1^*, x_2^*] = [-0.75, -0.618034\ ]$.



(a) (b) (c)

Figure 2.1: Feasible regions (green shaded) of the above Example 1, the NSIP constraint depending on $u \in [0,1]$, fig.(a) for u= 0, fig.(b) for u=0.5, and fig.(c) for u=1.

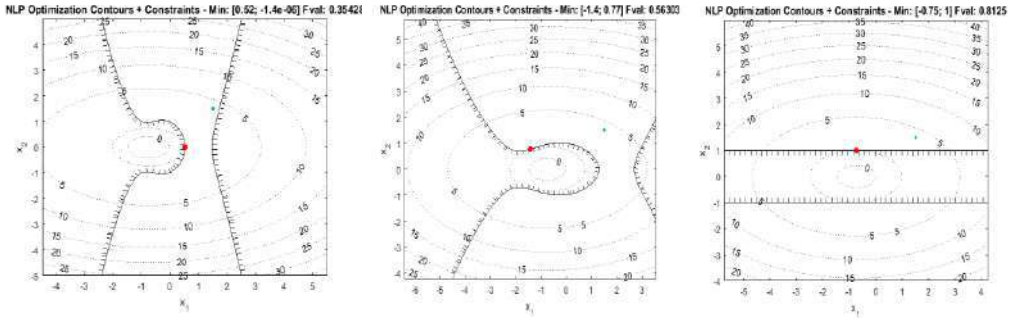


Figure 2.2: The optimal solution of the above NSIP problem found by using MATLAB depending on only three values of u (for u=1, for u=0.5 and for u=0 respectively).

2.1 Optimality conditions for NSIP:

The feasible set of the above problem (P) can be written as

$$S = \left\{ x \in X \middle| g(x, u^i) \leq 0, x \in u^i \in U \right\}$$

In this section we use the generalized convex functions. An important generalization of convex function termed as symbolically convex function was first introduced by Ewing[12]. Kaul and Kaur[13] defined symbolically quasi

convex and pseudo convex. We also follow Weir and Mond[14] , Yang[15] and Li [16] which is related with preinvex function.

Definition 1: Let $f : X \to R^n$ be any function, then we can say that the set $X \subset R^n$ is an η – locally starshaped set at $x, x^* \in X$ If for any $x \in X$ there exist $0 < a_\eta(x, x^*) \leq 1$ such that $x^* + \lambda\eta(x, x^*) \in X$ for any $\lambda \in [0, a_\eta(x, x^*)]$.

Definition 2 (Hayashi and komiya[17]): Let $f : X \to R^n$ be any function, then it said to be a convexlike function if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$ then there exist $y \in X$ such that $f(y) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$.

Definition 3: Let $f : X \to R^n$ be any function, where $X \subset R^n$ is an η – locally starshaped set at $x^* \in X$. Then we say that f is η – *semidiffer entiable* at any $x^* \in X$, if $\nabla^+ f(x^*, \eta(x, x^*))$ exist for each $x \in X$ such that

$$\nabla^+ f(x^*, \eta(x, x^*)) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[f(x^* + \lambda\eta(x, x^*)) - f(x^*)\right]$$

(The right derivative at x^* along the direction $\eta(x, x^*)$

And if f is semi differentiable at any $x^* \in X$, then f is said to be η – *semidiffer entiable* at x^* on X .

Theorem 1: Let $x^* \in X$, be a local weak minimum solution for the problem:

$$\text{Minimize } (f_1(x), ..., f_p(x)),$$

$$\text{subject to } g(x, u^i) \leq 0, i \in I$$

$$x \in X \subset S,$$

Here $U \subset R^m$ is an infinite countable set, $f = (f_1, f_2, ..., f_p) | X \to R^p$ and $g | X \times U \to R^m$ are η – *semidiffer entiable* at x^* . Also consider that $g(x, u^i)$, $i \in J(x^*)$ is a continuous function at x^* and $\nabla^+ f(x^*, \eta(x, x^*))$ and

$\nabla^+ g\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right)$ are convex like functions on X, where $I = \left\{i : g\left(x, u^i\right) \leq 0, x \in X, u^i \in U\right\}$, the set of i for which the active constraint is $I\left(x^*\right) = \left\{i \middle| g\left(x, u^i\right) = 0, x \in X, u^i \in U\right\}$ and $J\left(x^*\right) = \left\{i \middle| g\left(x, u^i\right) < 0, x \in X, u^i \in U\right\}$ if $g\left(x, u^i\right)$ satisfies a regularity condition at x^* (see[Definition 3.2][18] then there exist $\lambda_0 \in R^p$ and $\left(\lambda_i\right)_{i \in I} \in R^m$ such that

$$\lambda_0^T \nabla^+ f\left(x^*, \eta\left(x, x^*\right)\right) + \sum_{i \in I} \lambda_i \nabla^+ g\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) \geq 0 \text{ for all } x \in X,$$

$$\sum_{i \in I} \lambda_i g\left(x^*, u^i\right) = 0, g\left(x^*, u^i\right) \leq 0,$$

$$\lambda_0^T e = 1, \lambda_0 \geq 0, \left(\lambda_i\right)_{i \in I} \geq 0, \text{ where } e = (1, \ldots\ldots 1)^T \in R^p.$$

2.2 Necessary optimality conditions for NSIP:

Lemma 1: Let $x^* \in X \subset S$ be a local weak minimum solution for (P) and let there exist $x_1^* \in X$. Further we assume that $g\left(x, u^i\right)$ is continuous at x^* for any $i \in J\left(x^*\right)$and f, g^* are $\eta - semidiffer entiable$ at x^*. Then the system

$$\nabla^+ f\left(x^*, \eta\left(x, x^*\right)\right) < 0$$

$$\nabla^+ g^*\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) < 0, \tag{2.2}$$

has no solution at $x \in X$.

Proof: Let x^* be a local weak minimum solution for (P) and suppose that there exist $x_1^* \in X$ such that $\nabla^+ f\left(x^*, \eta\left(x_1^*, x^*\right)\right) < 0$ (2.3)

$$\nabla^+ g^*\left(\left(x^*, \eta\left(x_1^*, x^*\right)\right), u^i\right) < 0, \tag{2.4}$$

Now consider a function $\phi\left(x^*, x_1^*, \lambda\right) = f\left(x^* + \lambda\eta\left(x^*, x_1^*\right)\right) - f\left(x^*\right)$

Then we obviously get $\phi\left(x^*, x_1^*, 0\right) = 0$ and the right differential of $\phi\left(x^*, x_1^*, 0\right)$ with respect to λ at $\lambda = 0$ we get the following result and which is given by

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left[\phi\left(x^*, x_1^*, \lambda\right)-\phi\left(x^*, x_1^*, 0\right)\right]=\lim_{\lambda\to 0}\frac{1}{\lambda}\left[f\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right)-f\left(x^*\right)\right]$$

$$\Rightarrow \lim_{\lambda\to 0}\frac{1}{\lambda}\left[\phi\left(x^*, x_1^*, \lambda\right)-\phi\left(x^*, x_1^*, 0\right)\right]=\nabla^+ f\left(x^*, \eta\left(x_1^*, x^*\right)\right)$$

Using (2.3), we get $\lim_{\lambda\to 0}\frac{1}{\lambda}\left[\phi\left(x^*, x_1^*, \lambda\right)-\phi\left(x^*, x_1^*, 0\right)\right]<0.$

Therefore, $\phi\left(x^*, x_1^*, \lambda\right)<0$ if λ lies in some open interval, i.e. $\lambda\in\left(0,\delta_1\right)$, $\delta_1>0$ so we get

$$f\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right)<f\left(x^*\right), \text{ for } \lambda\in\left(0,\delta_1\right).$$

Similarly, we get another result $g^*\left(\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right), u^i\right)<g^*\left(x^*, u^i\right)=0$ for $\lambda\in\left(0,\delta_2\right)$, $\delta_2>0$. Now since $i\in J\left(x^*\right)$, $g\left(x, u^i\right)<0$ and $g\left(x, u^i\right)$ is continuous at x^*, so there exist another positive integer $\delta'>0$ such that $g\left(\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right), u^i\right)<0$, $\lambda\in\left(0,\delta'\right)$, for any $i\in J\left(x^*\right)$, Consider $\delta=\min\left(\delta_1,\delta_2,\delta'\right)$, then

$$x^*+\lambda\eta\left(x_1^*, x^*\right)\in S_\delta\left(x^*\right)\subset N_\delta\left(x^*\right),\ \lambda\in\left(0,\delta\right), \tag{2.5}$$

Where $S_\delta\left(x^*\right)$ is a hyperplane around x^* and $N_\delta\left(x^*\right)$ is a nbd. of x^*. So we obviously get

$$f\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right)<f\left(x^*\right), \tag{2.6}$$

$$g\left(\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right), u^i\right)<0, \text{ for any } \lambda\in\left(0,\delta\right) \tag{2.7}$$

Now from (2.5) and (2.7), we have

$$x^*+\lambda\eta\left(x_1^*, x^*\right)\in X\cap N_\delta\left(x^*\right), \text{ for any } \lambda\in\left(0,\delta\right).$$

Again using (2.6), for $B(x)=\left(f_1(x),\ldots\ldots, f_p(x)\right)$, we get the following result

$$B\left(x^*+\lambda\eta\left(x_1^*, x^*\right)\right)<B\left(x^*\right).$$

But this contradicts the assumptions that, x^* is a local weak minimum solution for (P). Hence there is no solution like $x \in X$ satisfying the system (2.2).

Proposition 1 (FJ type necessary conditions): In case of problem (P), let us consider that $g(x,u^i)$ is continuous at x^* for $i \in J(x^*)$, $\nabla^+ f(x^*, \eta(x,x^*))$ and $\nabla^+ g^*((x^*, \eta(x,x^*)),u^i)$ are convex like functions on X. If x^* is a local minimum solution for (P), then there exist $\lambda_0 \in R^p$ and $\lambda = (\lambda_i)_{i\in I} \in R^m$ such that

$$\lambda_0^T \nabla^+ f(x^*, \eta(x,x^*)) + \sum_{i\in I(x^*)} \lambda_i \nabla^+ g^*((x^*, \eta(x,x^*)),u^i) \geq 0 \text{ for all } x \in X, \quad (2.8)$$

$$\sum_{i\in I} \lambda_i g(x^*, u^i) = 0 \quad (2.9)$$

$$(\lambda_0, \lambda) \neq 0, (\lambda_0, \lambda) \geq 0. \quad (2.10)$$

Proof: If x^* is a (local) weak minimum solution for (P), then by Lemma 1, the system (2.2) has no solution at $x \in X$. But the assumptions hold (Hayashi and komiya[19]) also holds and since the system (2.2) has no solution at $x \in X$

Then we obviously get the following result, i.e. there exists $\lambda_0 \in R^p, \lambda_i \in R$ $(i \in I(x^*))$, such that $\lambda_0 \geq 0, \lambda_i \geq 0$ $(i \in I(x^*)), (\lambda_0, (\lambda_i)_{i\in I(x^*)}) \neq 0$ with

$$\lambda_0^T \nabla^+ f(x^*, \eta(x,x^*)) + \sum_{i\in I(x^*)} \lambda_i \nabla^+ g^*((x^*, \eta(x,x^*)),u^i) \geq 0 \text{ for all } x \in X, \quad (2.11)$$

Now if we put $\lambda_i = 0$ for $i \in J(x^*)$, by (2.11) we obviously get (2.8).

Finally, the relation (2.7) follows obviously and hence the proof.

Proposition 2: (KKT type necessary conditions): In case of problem (P), let us consider that $g(x,u^i)$ is continuous at x^* for $i \in J(x^*)$, $\nabla^+ f_j(x^*, \eta(x,x^*))$ $j = (1,2,...., p)$ and $\nabla^+ g^*((x^*, \eta(x,x^*)),u^i)$ are convex like functions on X. If g satisfies (GSCQ) at x^*.and If x^* is a local minimum solution for (P), then there exist $\lambda_0 \in R_+^p$ and $\lambda \in R^m$ such that

$$\sum_{j=1}^{p}(\lambda_0)_j \nabla^+ f_j\left(x^*, \eta\left(x, x^*\right)\right) + \sum_{i\in I} \lambda_i \nabla^+ g^*\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) \geq 0 \text{ for all } x \in X,$$

$$\sum_{i\in I} \lambda_i g\left(x^*, u^i\right) = 0 \quad g\left(x^*, u^i\right) \leq 0$$

$$\lambda_0^T e = 1, \lambda_0 \geq 0, \lambda \geq 0, \text{ where } e = (1, \ldots\ldots 1)^T \in R^p.$$

Proof: If x^* is a local weak minimum solution for (P), then applying Theorem 1 to the problem (P) we can write that there exist $\lambda_0 \in R_+^p$ and $\lambda \in R^m$ such that

$$\sum_{j=1}^{p}(\lambda_0)_j \nabla^+ f_j\left(x^*, \eta\left(x, x^*\right)\right) + \sum_{i\in I} \lambda_i \nabla^+ g^*\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) \geq 0 \text{ For all } x \in X, \quad (2.12)$$

$$\sum_{i\in I} \lambda_i g\left(x^*, u^i\right) = 0 \quad (2.13)$$

$$g\left(x^*, u^i\right) \leq 0 \quad (2.14)$$

$$\lambda_0^T e = 1, \quad (2.15)$$

$$\lambda_0 \geq 0, \lambda \geq 0 \quad (2.16)$$

Thus completes the proof.

2.3 Sufficient optimality conditions for NSIP

In this section we use the concept of (locally) weak optimality, and give some sufficient optimality results for the so called problem (P).

(Hayashi and komiya[17]) and we will follow the definitions of Mishra et al.[20]

Definition 4: In case of problem (P), let $f : X \to R^n$ be a function, where $X \subset R^n$ is an η – locally starshaped set at $x^* \in X$. Then we say that f is:

i. Semilocally preinvex (slpi) at x^* if, corresponding to x^* and each $x \in X$, there exist a positive number $d_\eta\left(x, x^*\right) \leq a_\eta\left(x, x^*\right)$ such that

$$f\left(x^* + \lambda\eta\left(x, x^*\right)\right) \leq \lambda f(x) + (1-\lambda) f\left(x^*\right) \text{ for } 0 < \lambda < d_\eta\left(x, x^*\right);$$

ii. Semilocally quasi-preinvex (slqpi) at x^* if, corresponding to x^* and each $x \in X$, there exist a positive number $d_\eta\left(x, x^*\right) \leq a_\eta\left(x, x^*\right)$ such that $f(x) \leq f\left(x^*\right)$ for $0 < \lambda < d_\eta\left(x, x^*\right)$; implies $f\left(x^* + \lambda\eta\left(x, x^*\right)\right) \leq f\left(x^*\right)$.

Definition 5: In case of problem (P), and this problem is $\eta - semidiffer entiable$ type I-preinvex at x^* if for any $x \in X$, then we get

$$f_j(x) - f_j\left(x^*\right) \geq \nabla^+ f_j\left(x^*, \eta\left(x, x^*\right)\right), \forall j \in J = \{1,2,..., p\},$$

$$-g\left(x^*, u^i\right) \geq \nabla^+ g\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right), \ i \in I$$

Definition 6: In case of problem (P), and this problem is $\eta - semidiffer entiable$ pseudo-quasi-type I-preinvex at x^* if for any $x \in X$, then we get

$$\nabla^+ f_j\left(x^*, \eta\left(x, x^*\right)\right) \geq 0 \Rightarrow f_j(x) \geq f_j\left(x^*\right), \forall j \in J$$

$$-g\left(x^*, u^i\right) \leq 0 \Rightarrow \nabla^+ g\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) \leq 0, \forall i \in I$$

The problem (P) is $\eta - semidiffer entiable$ pseudo-quasi-type I-preinvex on X if it is $\eta - semidiffer entiable$ pseudo-quasi-type I-preinvex at any $x^* \in X$.

Theorem 2: In case of problem (P), let us consider that $x^* \in X$ be η – symbolically type I-preinvex at x^*. Also we can assumed that there exist $\lambda_0 \in R^p$ and $\lambda = (\lambda_i)_{i \in I} \in R^m$ such that

$$\sum_{j=1}^{p} (\lambda_0)_j \nabla^+ f_j\left(x^*, \eta\left(x, x^*\right)\right) + \sum_{i \in I} \lambda_i \nabla^+ g^*\left(\left(x^*, \eta\left(x, x^*\right)\right), u^i\right) \geq 0 \text{ for all } x \in X, \quad (2.17)$$

$$\sum_{i \in I} \lambda_i g\left(x^*, u^i\right) = 0 \quad (2.18)$$

$$\lambda_0^T e = 1, \tag{2.19}$$

$$\lambda_0 \geq 0, \lambda \geq 0 \tag{2.20}$$

.

Then x^* is a weak minimum solution for the problem (P).

Proof: First of all we have to proceed by contradiction. Let us consider that $\widetilde{x} \in X$ such that

$$f_j(\widetilde{x}) < f_j(x^*) \text{ for any } j \in J \text{, where } J = \{1,2,..., p\}. \tag{2.21}$$

Since the problem (P) is η – symbolically type I-preinvex at x^*, then we get the following result

$$f_j(\widetilde{x}) - f_j(x^*) \geq \nabla^+ f_j\left(x^*, \eta(\widetilde{x}, x^*)\right), \forall j \in J = \{1,2,..., p\}, \tag{2.22}$$

$$-g(x^*, u^i) \geq \nabla^+ g\left(\left(x^*, \eta(\widetilde{x}, x^*)\right), u^i\right), \ i \in I \tag{2.23}$$

Now multiplying (2.22) by the condition $(\lambda_0)_j \geq 0, j \in J, \lambda_0 \in R^p$ and (2.23) by $\lambda_i \geq 0, i \in I$, and then summing the obtained result of the relations is given by

$$\sum_{j=1}^{p} (\lambda_0)_j \left(f_j(\widetilde{x}) - f_j(x^*)\right) - \sum_{i \in I} \lambda_i g(x^*, u^i)$$

$$\geq \sum_{j=1}^{p} (\lambda_0)_j \nabla^+ f_j\left(x^*, \eta(\widetilde{x}, x^*)\right) + \sum_{i \in I} \lambda_i \nabla^+ g\left(\left(x^*, \eta(\widetilde{x}, x^*)\right), u^i\right) \geq 0,$$

In the above relation we observe that the last inequality is according to the relation (2.17). So we obviously get the following result

$$\sum_{j=1}^{p} (\lambda_0)_j \left(f_j(\widetilde{x}) - f_j(x^*)\right) - \sum_{i \in I} \lambda_i g(x^*, u^i) \geq 0. \tag{2.24}$$

Since $x \in X$, $\lambda \geq 0$ so by (2.18) and (2.24) we obtain

$$\sum_{j=1}^{p} (\lambda_0)_j \left(f_j(\tilde{x}) - f_j(x^*) \right) \geq 0 . \tag{2.25}$$

Now using (2.19), (2.20) and (2.25) we get that there exist $j_0 \in J$ such that

$f_{j_0}(\tilde{x}) \geq f_{j_0}(x^*)$ But this is the contradiction to (2.21).

Thus the theorem is proved and we can easily state that x^* is a weak minimum solution for the problem (P).

Theorem 3: In case of problem (P), let us consider that $x^* \in X$, $\lambda_0 \in R^p$ and $\lambda \in R^m$ such that the conditions (2.17)-(2.20) of the Theorem 2 hold. Moreover let us assume that the problem (P) is $\eta - semidiffer\ entiable$ pseudo-quasi-type I-preinvex at x^*. Then x^* is a weak minimum solution for the problem (P).

Proof: First of all assume that x^* is not a weak minimum solution for the problem (P), then there exist $\tilde{x} \in X$ such that

$f_j(\tilde{x}) < f_j(x^*)$ for any $j \in J$, where $J = \{1,2,..., p\}$.

Now for the Definition 5 i.e. by the $\eta - semidiffer\ entiable$ pseudo-quasi-type I-preinvex of (P) at x^*, we immediately get the following result

$$\nabla^+ f_j(x^*, \eta(\tilde{x}, x^*)) < 0 \text{ for any } j \in J$$

Again using the conditions, $(\lambda_0)_j \in R_+^p$, $\lambda_0^T e = 1$ we obtain that

$$\sum (\lambda_0)_j \nabla^+ f_j(x^*, \eta(\tilde{x}, x^*)) < 0 . \tag{2.26}$$

Now for $i \in I(x^*)$ and $g(x^*, u^i) = 0$ and it follows that

$$-g(x^*, u^i) \leq 0 \text{ , for any } i \in I(x^*).$$

And for η – *semidiffer entiable* pseudo-quasi-type I-preinvex of (P) at x^*, we get the result

$$\nabla^+ g\left(\left(x^*, \eta\left(\tilde{x}, x^*\right)\right), u^i\right) \leq 0 \text{ , for any } i \in I\left(x^*\right).$$

Since, $\lambda \in R_+^m$ and, putting $\lambda_i = 0$ and for any $i \in I\left(x^*\right)$, we get

$$\sum_{i \in I} \lambda_i \nabla^+ g\left(\left(x^*, \eta\left(\tilde{x}, x^*\right)\right), u^i\right) \leq 0 \tag{2.27}$$

Now using (2.26) and (2.27) we obviously obtain

$$\sum_{j=1}^{p} \left(\lambda_0\right)_j \nabla^+ f_j\left(x^*, \eta\left(\tilde{x}, x^*\right)\right) + \sum_{i \in I} \lambda_i \nabla^+ g\left(\left(x^*, \eta\left(\tilde{x}, x^*\right)\right), u^i\right) < 0 \text{ ,}$$

But it is contradiction to (2.17).

Thus x^* is a weak minimum solution for the problem (P) and hence the theorem is proved.

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